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ON AZUMAYA GROUP RINGS

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Abstract

Let R be a ring with 1, R_0 the center of R, G a group, RG a group ring of G over R, and C the center of RG. If RG is Azumaya, then so is RK for every subgroup K of G. For a subgroup K of finite order $|K|$ invertible in R, if RG is Azumaya, then RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, which are direct summands of RG as bimodules over themselves, where $\overline{K}$ is the inner automorphism group of the group ring RG induced by the elements of K. Also, for any subgroup K of G, the converse holds.

1. Introduction

Let R be a ring with 1, R_0 the center of R, G a group, RG a group ring of G over R, and C the center of RG. In [2], it is shown that RG is an Azumaya algebra over C, if and only if there exists a subgroup H of G such that $G = ZH$, where Z is the center of G and RH is an Azumaya algebra ([2], Lemmas 2.2, 4.2, and 4.3). In the present paper, we shall show that if RG is Azumaya, then so is RK for each subgroup K of G.

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Thus, we give a different proof of the above characterization of an Azumaya group ring RG. Moreover, let K be any subgroup of finite order $|K|$ invertible in R. If RG is Azumaya, then RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, which are direct summands of RG as bimodules over themselves. This implies a characterization of an Azumaya group ring RG in terms of Hirata separable extensions.

2. Basic Definitions and Notations

Let B be a ring with 1, and A be a subring of B with the same identity 1. Then B is called a *separable extension of A*, if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, k$ for some integer $k\}$ such that $\sum a_i b_i = 1$, and $\sum x a_i \otimes b_i = \sum a_i \otimes b_i x$ for all x in B, where $\otimes$ is over A. In particular, B is called an *Azumaya algebra*, if it is a separable extension over its center. A ring B is called a *Hirata separable extension of A,* if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. For more about Azumaya algebras and Hirata separable extensions, see [5], [6], and [7]. The commutator subring of A in B is denoted by $V_B(A)$.

Throughout this paper, R will be a ring with identity 1, R_0 the center of R, G a group, RG a group ring of G over R, and C the center of RG.

3. Subgroup Rings

In this section, let RG be an Azumaya group ring. We shall show that for any subgroup K of G, RK is also Azumaya. Then, we derive a characterization of an Azumaya group ring RG by using subgroups K of G such that $G = ZK$, where Z is the center of G. We begin with an important characterization of an Azumaya group ring in [2].

Proposition 3.1 ([2], Theorem 1)**.** *The group ring RG is an Azumaya algebra, if and only if* (i) *R is an Azumaya algebra over* R_0, (ii) *the center*

Z of G has a finite index, *and* (iii) *the order of the commutator subgroup G' of G is a finite integer and invertible in R.*

Lemma 3.2. *Let K be a subgroup of G. If RG is Azumaya*, *then RK is Azumaya.*

Proof. Since RG is Azumaya, $|G/Z|$ is finite by Proposition 3.1. Hence, $|(KZ)/Z| \leq |G/Z| < \infty$. But $(KZ)/Z \cong K/(K \cap Z)$, so $|(KZ)/Z| = |K/(K \cap Z)| < \infty$. Let $Z(K)$ be the center of K. Then $K \cap Z \subset Z(K)$; and so $|K/Z(K)| \leq |K/(K \cap Z)| < \infty$. Moreover, the commutator subgroup K' of K is contained in G', so $|K'|$ is finite and invertible in R for $|G'|$ is finite and invertible in R by Proposition 3.1, again. Noting that R is an Azumaya algebra over R_0, we conclude that RK is Azumaya by Proposition 3.1.

Now, we give a different proof of the characterization of an Azumaya group ring RG by using subgroups K of G.

Theorem 3.3. *The group ring RG is Azumaya, if and only if for each subgroup K of G such that $G = ZK$, RK is Azumaya.*

Proof. By Lemma 3.2, the necessity is true. For the sufficiency, let $Z(K)$ be the center of K. Since, $G = ZK, Z(K) \subset Z$. Hence, $Z(K) = Z \cap K$. By hypothesis, RK is Azumaya, so $|K / Z(K)| = |K/(K \cap Z)| < \infty$ by Proposition 3.1. Noting that $G = ZK$, we have that $|G/Z| = |(ZK)/Z| = |K/(K \cap Z)| < \infty$. Moreover, since $G = ZK$ again, $G' = (ZK)' = K'$, so $|G'| = |K'|$, which is finite and invertible in R (for RK is an Azumaya algebra). Also R is Azumaya, so RG is Azumaya by Proposition 3.1.

4. Hirata Separable Extensions

Let K be a finite subgroup of G. We shall show that, if RG is Azumaya and $|K|$ is invertible in R, then RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, which are direct summands of RG as bimodules over themselves. This leads to a characterization of an Azumaya group ring RG in terms of Hirata separable extensions. We shall employ a well known property of a group ring of a finite group.

Lemma 4.1. *If K is a finite group and $|K|$ is invertible in R, then RK is a separable extension of R.*

Lemma 4.2. *Let RG be Azumaya. If K is a finite subgroup of G such that $|K|^{-1} \in R$, then* (i) $(RC)K$ *and* $(R_0G)^{\overline{K}}$ *are direct summands of RG as bimodules over themselves, where $\overline{K}$ is the inner automorphism group of the group ring RG induced by the elements of K, and* (ii) *RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively.*

Proof. (i) Since $|K|^{-1} \in R$, RK is a separable extension of R by Lemma 4.1. By hypothesis, RG is Azumaya, so R is an Azumaya algebra over R_0 by Proposition 3.1. Hence, RK is a separable R_0-algebra by the transitivity property of separable extensions. Thus, $C \otimes_{R_0} RK$ is a separable C-algebra; and so as a homomorphic image of $C \otimes_{R_0} RK$, $(CR)K$ is a separable C-algebra. Since RG is an Azumaya C-algebra, $V_{RG}((RC)K)$ is a separable C-subalgebra of RG by the commutator theorem for Azumaya algebras ([1], Theorem 4.3, page 57). Noting that $V_{RG}((RC)K) = V_{RG}(RK) = (R_0G)^{\overline{K}}$, we have that $(R_0G)^{\overline{K}}$ is a separable subalgebra of RG over C. But then, both $(RC)K$ and $(R_0G)^{\overline{K}}$ are direct summands of the Azumaya algebra RG as bimodules over themselves. This proves part (i). Moreover, RG is projective over $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, ([1], Proposition 2.3, page 48). Therefore, RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, ([3], Theorem 1). This proves part (ii).

To obtain a characterization of an Azumaya group ring RG in terms of Hirata separable extensions, we shall employ a result as given by Sugano.

Lemma 4.3 ([4], Proposition 1.3). *Let B be a Hirata separable extension of A and A is a direct summand of B as an A-bimodule. Then, $V_B(A)$ is a separable algebra over the center of B.*

Theorem 4.4. *A group ring RG is Azumaya, if and only if there exists a subgroup K of G such that* (i) *RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^{\overline{K}}$, respectively, and* (ii) *$(RC)K$ and $(R_0G)^{\overline{K}}$ are direct summands of RG as bimodules over themselves, where $\overline{K}$ is the inner automorphism group of the group ring RG induced by the elements of K.*

Proof. $(\Rightarrow)$ By taking G' as K in Lemma 4.2, the necessity is a consequence of Lemma 4.2 because $|G'| < \infty$ and $|G'|^{-1} \in R$ by Proposition 3.1.

$(\Leftarrow)$ Since, RG is a Hirata separable extension of $(RC)K$ and $(RC)K$ is a direct summand of RG as a bimodule over $(RC)K$, $V_{RG}((RC)K)$ is a separable subalgebra of RG over C by Lemma 4.3. Thus, $(R_0G)^{\overline{K}}$ $(= V_{RG}((RC)K))$ is a separable subalgebra of RG over C. By hypothesis, RG is a Hirata separable extension of $(R_0G)^{\overline{K}}$, so RG is a separable extension of $(R_0G)^{\overline{K}}$. Therefore, RG is a separable C-algebra by the transitivity property of separable extensions; and so RG is Azumaya.

We conclude this paper with three examples to demonstrate the results of the paper.

Example 1. Let G be a nonabelian group and R be the ring of integers. Then $|G'|$ is not invertible in R; and so RG is not Azumaya.

Example 2. Let G be a finite nonabelian group, R be the ring of integers, and $|G'| = m$. Then R_mG is Azumaya, where R_m is the ring of R localized with respect to the multiplicatively closed set $\{1, m, m^2, m^3, \cdots\}$ because $|G'|$ is invertible in R_m.

Example 3. Let $R_m G$ be given in Example 2 and K be a subgroup of G such that $K \subset G'$. Then (1) $R_m K$ is Azumaya by Lemma 3.2, (2) $R_m G$ is a Hirata separable extension of $(R_m C)K$ and $(R_m G)^{\overline{K}}$, respectively, and $(R_m C)K$ and $(R_m G)^{\overline{K}}$ are direct summands of $R_m G$ as bimodules over themselves by Theorem 4.4.

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